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Transient power variation in surface conditions in heat conduction for plates

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Abstract

Transient temperature solutions in plates are derived for heating conditions varying as time to an integer power at a surface. The powers include 0, 1, 2 and 3; the last of which is useful for cubic splines. Boundary conditions of the first, second and third kinds are treated at both surfaces with the non-homogeneity at $x = 0$. As the power increases by one, an additional quasi-steady summation term appears in the analytical solution. Algebraic forms for these summations are derived in a systematic way. Extensive tables are given along with an example.

Keywords: Analytical solutions; Heat conduction; Plates; Transient boundary conditions

1. Introduction

Realistic applications in heat conduction usually have transient variations at the boundaries. Many examples can be cited such as in the ignition of a rocket engine, heating of brake drums of cars and trucks during braking, and cooking of foods. However, few exact solutions for transient boundary conditions exist. Most advanced heat conduction books instead concentrate on exact solutions for constant boundary conditions and recommend using those solutions with Duhamel's integral [1] to obtain solutions for time-variable boundary conditions. Other methods, such as employing Green's functions, are also recommended [2]. For one-dimensional problems in plates, the solutions for constant boundary conditions can be obtained using the methods of the separation of variables (SOV) or the Laplace transform [1,2]. The SOV solutions tend to be more convenient to manipulate because the temporal and spatial dependences are independent, unlike those found using Laplace transforms. In the SOV solution of heat conduction problems with heat flux (or other) boundary conditions, several time-independent but space-dependent summations may be obtained, some of which may converge very slowly. Instead of these infinite-series summations, algebraic identities, if available, are preferred. However, these identities are not directly obtained using either Duhamel's integral or using Green's functions. The primary objective of this paper is to provide a systematic method for determining these algebraic identities for the steady- or quasi-steady state summations that arise from boundary conditions varying as time to an integer power; the integers considered here are 0, 1, 2 and 3.

A number of methods of eliminating the slowly converging series are possible. Some of these are given in this paper. By removing these summations, more insight into the solution is possible. (Ref. [3] gives methods to simplify summations in 2D heat-conduction problems with constant boundary conditions. See also [4].) Another motivation to develop these solutions is for approximating time-variable conditions with cubic *B*-splines [5], which are used in the solution of the inverse heat conduction problem [6].

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Nomenclature

$A_{XIJ,m}$	constant associated with mth eigenfunction
B	boundary condition modifier in numbering system
Bi_j	Biot number ($=h_jL/k$, $j=1, 2$)
$B_p(x)$	Bernoulli polynomial of order p
$C^{(i)}_{XIJ,m}$	constant of integration
$E_p(x)$	Euler polynomial of order p
G_{XIJ}	Green's function for rectangular plate with I and J boundary conditions
h_i	heat transfer coefficient on ith surface
i^merfc	mth integral of the complementary error function
I	indication of boundary condition at $x=0$, 1 – first kind, 2 – second kind, 3 – third kind
$IIS_{XIJ,i}$	twice integrated summation of eigenfunctions (defined in Eq. (20c))
J	indication of boundary condition at $x=1$, 1 – first kind, 2 – second kind, 3 – third kind
k	thermal conductivity
L	thickness of slab
N_m	mth norm of $X_m(x)$
n	exponent on time for boundary condition (i.e., $T\propto t^n$ and $q\propto t^n$)
q_0	boundary heat flux
t	time
t_N	characteristic time
$S^{(i)}_{XIJ}$	summation of eigenfunctions (Defined in Eqs. (15a) and (15e))
T	scaled temperature
$\widehat{T}$	temperature
$\widehat{T}_\infty$	ambient temperature
u	cotime ($=t-\tau$)
X	eigenfunction and notation for Cartesian coordinate
$X_{XIJ,m}(x)$	mth eigenfunction
x	scaled length ($=\hat{x}/L$)
$\hat{x}$	length

Greek letters

α	thermal diffusivity
$\beta_{XIJ,m}$	mth eigenvalue
Δ_{XIJ}	constant of integration ($\Delta_{X22}=1$, $\Delta_{XIJ}=0$ for other values of I and J)
$\Gamma(x)$	Gamma function
τ	dummy time variable of integration

The many solutions given herein have varied uses. One is to provide solutions that can be used in verification, that is, to provide solutions with which the accuracy of approximate methods, such as finite difference and finite element solutions, can be investigated [7–11]. Another is to provide relatively simple forms of exact solutions that can be used in estimating parameters [12,13].

Many different transient heat conduction problems are discussed in this paper. In order to identify clearly the different cases, a numbering system [2] is briefly described and then used. For 1D Cartesian problems the symbol starts with X followed by two numbers, the first is for the boundary condition at $\hat{x}=0$ and the second for the boundary condition at $\hat{x}=L$. A prescribed temperature, $\widehat{T}$, at a boundary is denoted by a 1. A prescribed heat flux, q, at a boundary is denoted by a 2. A prescribed ambient temperature, $\widehat{T}_\infty$, at a boundary is denoted by a 3. For example, a plate with a prescribed heat flux at $\hat{x}=0$ and a temperature condition at $\hat{x}=L$ is denoted $X21$. An additional type of boundary condition is indicated by a 0; a semi-finite body with a prescribed temperature at $\hat{x}=0$ is denoted $X10$.

Boundary condition modifiers are used to describe the boundary conditions more fully; this is done by following the $X21$, for example, by $B10$, where the B denotes boundary modifier, the 1 denotes a steady boundary condition at $\hat{x}=0$, and the 0 denotes a homogeneous boundary condition at $\hat{x}=L$. In this paper, the $\hat{x}=0$ conditions for $\widehat{T}$, q or $\widehat{T}_\infty$ vary as $\hat{t}^n$, where $n=0$, 1, 2 or 3. For the $n=1$ condition, a 2 follows B. For higher degree variations such as quadratic or cubic, $3n$ follows B with ($n=2$ or 3) given at the end of the notation. Finally, the initial condition is denoted with a T followed by a 0 for a zero initial condition or a 1 for a constant, non-zero initial condition. Thus, for example, the notation $X21B10T0$ denotes a plate with a uniform (non-zero) heat flux at $\hat{x}=0$ and zero temperature at $\hat{x}=L$ and an initial temperature of zero. Also, a general series of cases with $n=0$, 1, 2 and 3 is denoted $XIJB3n0T0$ ($n=0$, 1, 2, 3).

The literature is limited in regards to transient heat conduction in finite plates with boundary conditions that vary as a power of time. Possibly the greatest resource for solutions is given in Carslaw and Jaeger [14]. Page 63 of [14] gives solutions for the $X10B3nT0$ ($n=1/2, 1, 3/2,\dots$) cases, where the surface temperature is $\widehat{T}_0(\hat{t}/\hat{t}_1)^n$. On page 77 solutions are given for a semi-infinite body for a heat flux given by $-k\partial\widehat{T}/\partial\hat{x}(0,\hat{t})=\hat{q}_0(\hat{t}/\hat{t}_1)^n$ for $n=-1/2$, 0, 1/2, 1,… where k is thermal conductivity. For these values of n, solutions are given by

$$\widehat{T}(\hat{x},\hat{t})=\frac{\hat{q}_0}{k}2^{2n+1}\Gamma(n+1)\left(\frac{\hat{t}}{\hat{t}_1}\right)^n(\alpha\hat{t})^{1/2}i^{2n+1}\mathrm{erfc}\left(\frac{\hat{x}}{\sqrt{4\alpha\hat{t}}}\right) \tag{1a}$$

where α is the thermal diffusivity. At $\hat{x}=0$, the temperatures are

$$\widehat{T}(0,\hat{t})=\frac{q_0}{k}\frac{\Gamma(n+1)}{\Gamma(n+3/2)}\left(\frac{\hat{t}}{\hat{t}_1}\right)^n(\alpha\hat{t})^{1/2} \tag{1b}$$

On pages 113 and 114 of [14], plate solutions are given for a time-variable heat flux at $\hat{x} = L$; the notations are $X22B03nT0$ $(n = -1/2, 0, 1/2, 1, \ldots)$ and $X12B03nT0$ $(n = -1/2, 0, 1/2, 1, \ldots)$, where the first case is insulated at $\hat{x} = 0$ and the second case has a zero temperature at $\hat{x} = 0$; in both cases the heat flux at $\hat{x} = L$ is given by $k\partial\widehat{T}/\partial\hat{x}(L,\hat{t}) = q_0(\hat{t}/\hat{t}_1)^n$ and both solutions are given in terms of error functions. Solutions for plates in terms of error functions require more terms as time is increased and do not yield steady or quasi-steady components in the solutions. Carslaw and Jaeger [14, p. 127] have a solution for a linearly increasing $\widehat{T}_\infty$ (denoted by $X23B02T0$) that has an algebraic form for a quasi-steady term rather than a series form.

Ozisik [1] solves several problems having a linear time variation of the boundary condition. On page 208, Eq. (5-47) is a SOV-type solution for the $X22B20T0$ problem which is obtained using Duhamel's theorem; it has two series, one of which converges slowly. These series can be replaced by algebraic forms given herein. On pages 516–518, Ozisik [1] considers the $X21B02T0$ problem that contains two summations that are given algebraic expressions. One summation is evaluated on page 565 using the solution of the related $X21B00T1$ problem. The other summation is evaluated using a quadratic initial temperature.

Myers [15] discusses some problems with a boundary condition varying linearly with time. A solution to the $X11B20T0$ problem based on Duhamel's theorem is given [15, p. 161–163]. Two different forms of the solution are given, one of which has a single series that can be replaced by an algebraic form and the other has two such series. Myers [15, p. 112 and 113] shows how the quasi-steady series in the $X22B10T0$ problem can be replaced by an algebraic form.

Luikov [16] treats the problem of a linearly varying ambient temperature problem ($X23B02T0$) in two different ways. On pages 300–303, starting with the Laplace transform and then taking the inverse Laplace transform in a form suitable for large times, an algebraic form is directly obtained for the quasi-steady component of the solution. However, when the same problem is solved on page 347 using Duhamel's theorem, a quasi-steady state series is obtained, for which Luikov gives an algebraic expression and states that "a special proof is needed" to obtain it.

Polyanin [17] provides many solutions, including 2D and 3D geometries, in symbolic form using Green's functions. However, the solutions are in terms of summations that may converge slowly. Many of these summations for 1D heat conduction in plates with integer-power variations with respect to time can be expressed in terms of the algebraic forms given herein.

Although time-variable boundary conditions are treated in the references discussed above, few solutions are given and a general method is not given by which to replace the "steady-state or quasi-steady" series with algebraic forms. No explicit solutions are given for variations of the ambient temperature greater than the first power of time. For power variation of the heat flux at a surface, solutions are given for many different powers, but the solutions [14] are in terms of error functions. These error function solutions require more terms for evaluation as time increases and do not exhibit the underlying quasi-steady state behaviors.

The objective of this paper is to present a general method for deriving algebraic relations for the summation of some series. The method is demonstrated for boundary conditions at $\hat{x} = 0$ that vary as integer powers of time ($XIJB3n0T0$ ($n = 0$, 1, 2 or 3)) for linear ($n = 1$), quadratic ($n = 2$) and cubic ($n = 3$) variations. The boundary conditions of the first kind (I or $J = 1$), second kind (I or $J = 2$) and third kind (I or $J = 3$) are treated.

Section 2 presents the problem statement, while Section 3 gives a general solution. Section 4 provides solutions for particular boundary conditions and Section 5 presents some numerical and graphical results, followed by Section 6, with a summary and some conclusions.

2. Problem statement

Consider a transient, 1D problem in a plate which has a heat flux proportional to $\hat{t}^n$ for $n = 0$, 1, 2 and 3 time at $\hat{x} = 0$ and insulated at $\hat{x} = L$. The problem is denoted $X22B3n0T0$ ($n = 0$, 1, 2 and 3) and is described mathematically as

$$\frac{\partial^2 \widehat{T}^{(n)}}{\partial \hat{x}^2} = \frac{1}{\alpha}\frac{\partial \widehat{T}^{(n)}}{\partial \hat{t}}, \quad 0 < \hat{x} < L, \quad \hat{t} > 0 \tag{2}$$

$$-k\frac{\partial \widehat{T}^{(n)}}{\partial \hat{x}}(0,\hat{t}) = q_0\left(\frac{\hat{t}}{\hat{t}_N}\right)^n, \quad n = 0,1,2 \quad \text{or } 3 \tag{3a}$$

$$\frac{\partial \widehat{T}^{(n)}}{\partial \hat{x}}(L,\hat{t}) = 0 \tag{3b}$$

$$\widehat{T}^{(n)}(\hat{x},0) = \widehat{T}_i \tag{3c}$$

The superscript on the temperature denotes that the solution is for the power of n in the boundary condition at $\hat{x} = 0$ which is given by Eq. (3a). The time $\hat{t}_N$ is an arbitrary nominal time chosen to make the solution dimensionless.

For clarity only one set of boundary conditions are displayed in Eqs. (3a) and (3b), but three different conditions (first, second and third kinds) are possible both at $\hat{x} = 0$ and $\hat{x} = \widehat{L}$. A total of nine cases explicitly treated herein but others can be found by letting $\hat{x} \to L - \hat{x}$. The surface temperature at $\hat{x} = 0$ can be proportional to an integer power of time and the temperature rise of $\widehat{T}_s - \widehat{T}_i$ such as

$$\widehat{T}(0,\hat{t}) = (\widehat{T}_s - \widehat{T}_i)\left(\frac{\hat{t}}{\hat{t}_N}\right)^n + \widehat{T}_i \tag{4a}$$

The boundary convective condition has an ambient temperature that varies as $\hat{t}^n$

$$-k\frac{\partial \widehat{T}}{\partial \hat{x}}(0,\hat{t}) = h_1\left((\widehat{T}_\infty - \widehat{T}_i)\left(\frac{\hat{t}}{\hat{t}_N}\right)^n - (\widehat{T}(0,\hat{t}) - \widehat{T}_i)\right) \tag{4b}$$

The three boundary conditions at $\hat{x} = 0$ are then given by Eqs. (3b), (4a) and (4b). Analogous conditions to Eqs. (4a) and (4b) for $\hat{x} = L$ are

$$\widehat{T}(L, \hat{t}) = \widehat{T}_i \tag{5a}$$

$$-k\frac{\partial \widehat{T}}{\partial \hat{x}}(L, \hat{t}) = h_2\left(\widehat{T}(L, \hat{t}) - \widehat{T}_i\right) \tag{5b}$$

By subtracting the initial temperature $\widehat{T}_i$, Eqs. (5a) and (5b) become homogeneous.

For conciseness in the development below, the following dimensionless quantities are defined for the heat flux boundary condition problem described by Eqs. (2) and (3a)–(3c),

$$x = \frac{\hat{x}}{L} \tag{6a}$$

$$t = \frac{\alpha \hat{t}}{L^2} \tag{6b}$$

$$T_{X2J}^{(n)}(x, t) = \frac{\widehat{T}^{(n)}(\hat{x}, \hat{t}) - \widehat{T}_i}{(q_0 L/k)(L^2/(\alpha \hat{t}_1))^n} \tag{6c}$$

Then Eqs. (2) and (3a)–(3c) become (with the subscript $X2J$ on the temperature omitted)

$$\frac{\partial^2 T^{(n)}}{\partial x^2} = \frac{\partial T^{(n)}}{\partial t}, \quad t > 0, \quad 0 < x < 1 \tag{6d}$$

$$-\frac{\partial T^{(n)}}{\partial x}(0, t) = t^n \tag{6e}$$

$$\frac{\partial T^{(n)}}{\partial x}(1, t) = 0 \tag{6f}$$

$$T^{(n)}(x, 0) = 0 \tag{6g}$$

For boundary conditions of the first and third kinds at $x = 0$, the dimensionless temperatures are defined respectively as

$$T_{X1J}^{(n)}(x, t) \equiv \frac{\widehat{T}(\hat{x}, \hat{t}) - \widehat{T}_i}{\left(\widehat{T}_s - \widehat{T}_i\right)(L^2/\alpha \hat{t}_N)^n} \tag{7a}$$

$$T_{X3J}^{(n)}(x, t) \equiv \frac{\widehat{T}(\hat{x}, \hat{t}) - \widehat{T}_i}{(\widehat{T}_\infty - \widehat{T}_i)(L^2/\alpha \hat{t}_N)^n} \tag{7b}$$

(The subscripts on the temperature may be omitted in some of the following.) Then the dimensionless boundary conditions for the first and third kinds at $x = 0$ are

$$T_{X1J}^{(n)}(0, t) = t^n \tag{8a}$$

$$-\frac{\partial T_{X3J}^{(n)}}{\partial x}(0, t) = Bi_1(t^n - T_{X3J}^{(n)}(0, t)) \tag{8b}$$

where the Biot number is $Bi_1 = h_1 L/k$; for $x = 1$, the dimensionless boundary conditions are

$$T_{XI1}^{(n)}(1, t) = 0 \tag{9a}$$

$$-\frac{\partial T_{XI3}^{(n)}}{\partial x}(1, t) = Bi_2 T_{XI3}^{(n)}(1, t) \tag{9b}$$

where $Bi_2 = h_2 L/k$.

3. General solution for XIJ series of problems

A solution of the problem $X22B3n0T0$, (given by Eqs. (6d)–(6g)) using Green's functions is (or similarly using Duhamel's integral)

$$T_{X22}^{(n)}(x, t) = \int_{\tau=0}^{t} \tau^n G_{X22}(x, 0, t - \tau)\mathrm{d}\tau, \quad n = 0, 1, 2 \text{ or } 3 \tag{10a}$$

where the boundary condition at $x = 0$ varies as a power of time. The boundary condition at $x = 0$ is the second kind (given heat flux). For the boundary condition of the third kind at $x = 0$ and Jth kind at $x = 1$, the solution in terms of Green's functions, denoted $X3JB3n0T0$, can be written as

$$T_{X3J}^{(n)}(x, t) = Bi_1 \int_{\tau=0}^{t} \tau^n G_{X3J}(x, 0, t - \tau)\mathrm{d}\tau \tag{10b}$$

The solution given by Eq. (10a) can be modified to treat the boundary condition of the first kind (denoted $X1JB3n0T0$) by replacing

$$G_{X22}(x, 0, t - \tau) \text{ by } \frac{\partial G_{X12}}{\partial x'}(x, 0, t - \tau) \tag{11}$$

(The notation for the Green's function is $G(x, x', t - \tau)$, where the point of interest is at (x, t) and the instantaneous source is at (x', τ).) Notice the "1" subscript on the second G; it denotes a boundary condition of the first kind.

Using the long cotime (which we define as $t - \tau$) form of the transient heat conduction Green's function in a plate, we can write [2]

$$G_{XIJ}(x, x', t - \tau) = \Delta_{XIJ} + \sum_{m=1}^{\infty} \frac{X_{XIJ,m}(x)X_{XIJ,m}(x')}{N_{XIJ,m}} \mathrm{e}^{-\beta_{XIJ,m}^2(t-\tau)} \tag{12}$$

where $X_{XIJ,\,m}(x)$, $\beta_{XIJ,m}$ and $N_{XIJ,m}$ are the eigenfunctions, eigenvalues and norms, respectively. Eq. (12) is scaled and can be obtained by letting the dimensional Green's function have a thermal diffusivity and length L both equal to 1. If the boundary conditions are both of the second kind at both $x = 0$ and 1 (that is, the $X22$ case), then $\Delta_{X22} = 1$, otherwise $\Delta_{XIJ} = 0$, for $I = J \neq 2$. Restricting the boundary condition at $x = 0$ to be of the second kind (or third kind if Bi_1 is inserted before the integral sign as in Eq. (10b)) and introducing Eq. (12) into a general form of Eq. (10a) gives

$$T_{XIJ}^{(n)}(x, t) = \Delta_{XIJ}\frac{t^{n+1}}{n+1} + \sum_{m=1}^{\infty} \frac{X_{XIJ,m}(x)X_{XIJ,m}(0)}{N_{XIJ,m}} \times \int_{\tau=0}^{t} \tau^n \mathrm{e}^{-\beta_{XIJ,m}^2(t-\tau)}\mathrm{d}\tau, \quad I = 2 \text{ or } 3 \tag{13}$$

The integrals over τ for $n = 0$, 1, 2 and 3 are

$$\int_{\tau=0}^{t} e^{-\beta_{XIJ,m}^2(t-\tau)}d\tau = \frac{1}{\beta_{XIJ,m}^2} - \frac{e^{-\beta_{XIJ,m}^2 t}}{\beta_{XIJ,m}^2} \quad (14a)$$

$$\int_{\tau=0}^{t} \tau e^{-\beta_{XIJ,m}^2(t-\tau)}d\tau = \frac{1}{\beta_{XIJ,m}^2}t - \frac{1}{\beta_{XIJ,m}^4} + \frac{e^{-\beta_{XIJ,m}^2 t}}{\beta_{XIJ,m}^4} \quad (14b)$$

$$\int_{\tau=0}^{t} \tau^2 e^{-\beta_{XIJ,m}^2(t-\tau)}d\tau = \frac{1}{\beta_{XIJ,m}^2}t^2 - \frac{2}{\beta_{XIJ,m}^4}t + \frac{2}{\beta_{XIJ,m}^6} - \frac{e^{-\beta_{XIJ,m}^2 t}}{\beta_{XIJ,m}^6} \quad (14c)$$

$$\int_{\tau=0}^{t} \tau^3 e^{-\beta_{XIJ,m}^2(t-\tau)}d\tau = \frac{1}{\beta_{XIJ,m}^2}t^3 - \frac{3}{\beta_{XIJ,m}^4}t^2 + \frac{6}{\beta_{XIJ,m}^6}t - \frac{6}{\beta_{XIJ,m}^8} + \frac{e^{-\beta_{XIJ,m}^2 t}}{\beta_{XIJ,m}^8} \quad (14d)$$

This type of integral is present for boundary conditions of the first, second or third kinds at the $x = 0$ boundary. Notice that the repetition of $1/\beta_{XIJ,m}^2$ and similar terms in these equations; these terms are proportional to $1/\beta_{XIJ,m}^{2(i+1)}$ for $i = 0$, 1, 2 and 3. The last term for each of Eqs. (14a)–(14d) is a decaying exponential. The second to last term contributes a steady-state component while the other terms contribute to the temperature increase as a power of time.

The primary objective is to replace each summation in Eq. (13) and implied by Eqs. (14a)–(14d) with an algebraic form. The first summation implied in Eqs. (14a)–(14d) can be written (with the t-dependence omitted) as

$$S_{XIJ}^{(0)}(x) = \sum_{m=1}^{\infty} A_{XIJ,m}X_{XIJ,m}(x) \quad (15a)$$

where the coefficient $A_{XIJ,m}$ is determined by comparison of Eqs. (13) and (15a). For example, for the boundary condition of the second kind at $x = 0$, $X_{X2J,m}(x) = \cos(\beta_{X2J,m}x)$ (see Table 1, sixth row) and thus $A_{X2J,m}$ is given by (with a division by 2 added for convenience, since many norms are equal to 1/2)

$$A_{X2J,m} \equiv \frac{X_{X2J,m}(0)}{2N_{X2J,m}\beta_{X2J,m}^2} = \frac{\cos(0)}{2N_{X2J,m}\beta_{X2J,m}^2} = \frac{1}{2N_{X2J,m}\beta_{X2J,m}^2} \quad (15b)$$

For boundary conditions of the 1st kind at $x = 0$, $X_{X1J,m}(x) = \sin(\beta_{X1J}x)$ and $A_{X1J,m}$ is defined by

$$A_{X1J,m} \equiv \frac{dX_{X1J,m}(x')/dx'|_{x'=0}}{2N_{X1J,m}\beta_{X1J,m}^2} = \frac{1}{2N_{X1J,m}\beta_{X1J,m}} \quad (15c)$$

For boundary conditions of the third kind at both boundaries, $X_{X33,m}(x) = Bi_1\sin(\beta_{X33}x) + \beta_{X33}\cos(\beta_{X33}x)$ and A_m is defined by

$$A_{X33,m} \equiv \frac{Bi_1 X_{X33,m}(0)}{2N_{X33,m}\beta_{X33,m}^2} = \frac{Bi_1}{2N_{X33,m}\beta_{X33,m}} = \frac{Bi_1/\beta_{X33,m}}{(\beta_{X33,m}^2 + Bi_1)[1 + Bi_2/(\beta_{X33,m}^2 + Bi_2)] + Bi_1} \quad (15d)$$

A more general form of the summation than in Eq. (15a) is given by

$$S_{XIJ}^{(i)}(x) = \sum_{m=1}^{\infty} A_{XIJ,m}\frac{X_{XIJ,m}(x)}{\beta_{XIJ,m}^{2i}}, \quad i = 0,1,2 \text{ or } 3 \quad (15e)$$

The boundary conditions can be of the first, second or third kinds at either boundary (that is, I and $J = 1$, 2 or 3), but the solutions are restricted herein to non-homogeneous conditions only at $x = 0$.

Using the S functions just defined and Eqs. (14a)–(14d) in Eq. (13), the temperatures for $I, J = 1$, 2 and 3 for the t^n boundary conditions (for $n = 0$, 1, 2, and 3) are given by

$$T_{XIJ}^{(0)}(x,t) = \Delta_{XIJ}t + 2S_{XIJ}^{(0)}(x) - 2\sum_{m=1}^{\infty} e^{-\beta_{XIJ,m}^2 t}\frac{A_{XIJ,m}X_{XIJ,m}(x)}{\beta_{XIJ,m}^0} \quad (16a)$$

$$T^{(1)}(x,t) = \Delta_{XIJ}\frac{t^2}{2} + 2tS_{XIJ}^{(0)}(x) - 2S_{XIJ}^{(1)}(x) + 2\sum_{m=1}^{\infty} e^{-\beta_{XIJ,m}^2 t}\frac{A_{XIJ,m}X_{XIJ,m}(x)}{\beta_{XIJ,m}^2} \quad (16b)$$

$$T^{(2)}(x,t) = \Delta_{XIJ}\frac{t^3}{3} + 2t^2S_{XIJ}^{(0)}(x) - 4tS_{XIJ}^{(1)}(x) + 4S_{XIJ}^{(2)}(x) - 4\sum_{m=1}^{\infty} e^{-\beta_{XIJ,m}^2 t}\frac{A_{XIJ,m}X_{XIJ,m}(x)}{\beta_{XIJ,m}^4} \quad (16c)$$

$$T^{(3)}(x,t) = \Delta_{XIJ}\frac{t^4}{4} + 2t^3S_{XIJ}^{(0)}(x) - 6t^2S_{XIJ}^{(1)}(x) + 12tS_{XIJ}^{(2)}(x) - 12S_{XIJ}^{(3)}(x) + 12\sum_{m=1}^{\infty} e^{-\beta_{XIJ,m}^2 t}\frac{A_{XIJ,m}X_{XIJ,m}(x)}{\beta_{XIJ,m}^6} \quad (16d)$$

Notice that the S functions in a given equation are repeated in the next equation and another S function is added. For example, $S_{XIJ}^{(0)}(x)$ is present in each of these equations and $S_{XIJ}^{(1)}(x)$ is present in Eqs. (16b)–(16d). Unique S functions are found for each set of IJ values.

Another observation is that the derivative with respect to time of the temperature is related to the temperature by

$$\frac{\partial T^{(n)}}{\partial t}(x,t) = nT^{n-1}(x,t) \quad (17a)$$

which leads to the solution

$$T_{XIJ}^{(n)}(x,t) = n\int T_{XIJ}^{(n-1)}(x,t)dt + (-1)^{n-1}n!S_{XIJ}^{(n)}, \quad n = 1,2,\ldots \quad (17b)$$

These last two equations give a relation between the temperatures for successive powers, and apply for the nine cases in XIJ, I, $J = 1$, 2 or 3 and the four n values, for a total of 36 solutions. Eq. (17b) suggests a recursion relation between the successive solutions.

For large times and excluding the $X22$ case, Eq. (16a) becomes

$$T_{XIJ}^{(0)}(x,t)\Big|_{t \text{ large}} = T_{XIJ}^{(0)}(x) = 2S_{XIJ}^{(0)}(x) \quad (18a)$$

Table 1
Eigenfunctions, eigenvalues and the $A_{XIJ,m}$ function for I and $J = 1$, 2 and 3

I		$J=1$	$J=2$	$J=3$
1	$\beta_{X1J,m}$	$\beta_{X11,m} = m\pi$	$\beta_{X12,m} = \left(m - \frac{1}{2}\right)\pi$	eigencondition: $\beta_{X13,m} \cot \beta_{X13,m} = -Bi_2$
1	$X_{X1J,m}(x)$	$X_{X11,m}(x) = \sin\left(\beta_{X11,m} x\right)$	$X_{X12,m}(x) = \sin\left(\beta_{X12,m} x\right)$	$X_{X13,m}(x) = \sin\left(\beta_{X13,m} x\right)$
1	$A_{X1J,m}$	$A_{X11,m} = \frac{1}{\beta_{X11,m}}$	$A_{X12,m} = \frac{1}{\beta_{X12,m}}$	$\frac{1}{\beta_{X13,m}} \frac{\beta^2_{X13,m} + Bi_2^2}{\beta^2_{X13,m} + Bi_2^2 + Bi_2}$
2	$\beta_{X2J,m}$ *	$\beta_{X21,m} = \left(m - \frac{1}{2}\right)\pi$	$\beta_{X22,m} = m\pi$	eigencondition: $\beta_{X23,m} \tan \beta_{X23,m} = Bi_2$
2	$X_{X2J,m}(x)$	$X_{X21,m}(x) = \cos\left(\beta_{X21,m} x\right)$	$X_{X22,m}(x) = \cos\left(\beta_{X22,m} x\right)$	$X_{X23,m}(x) = \cos\left(\beta_{X23,m} x\right)$
2	$A_{X2J,m}$	$A_{X21,m} = \frac{1}{\beta^2_{X21,m}}$	$\frac{1}{\beta^2_{X22,m}}, m = 1, 2, ..$	$\frac{1}{\beta^2_{X23,m}} \frac{\beta^2_{X23,m} + Bi_2^2}{\beta^2_{X23,m} + Bi_2^2 + Bi_2}$
3	$\beta_{X3J,m}$	eigencondition: $\beta_{X31,m} \cot \beta_{X31,m} = -Bi_1$	eigencondition: $\beta_{X32,m} \tan \beta_{X32,m} = Bi_1$	eigencondition: $\tan\left(\beta_{X33,m}\right) = \frac{\beta_{X33,m}(Bi_1 + Bi_2)}{\beta^2_{X33,m} - Bi_1 Bi_2}$
3	$X_{X3J,m}(x)$	$\sin\left(\beta_{X31,m}(1-x)\right)$	$\cos\left(\beta_{X32,m}(1-x)\right)$	$Bi_1 \sin(\beta_{X33,m} x) + \beta_{X33,m} \cos(\beta_{X33,m} x)$
3	$A_{X3J,m}$	$\frac{Bi_1 \sin(\beta_{X31,m})}{\beta^2_{X31,m}\left(1 + \frac{Bi_1}{\beta^2_{X31,m} + Bi_1^2}\right)}$	$\frac{Bi_1 \cos(\beta_{X32,m})}{\beta^2_{X32,m}\left(1 + \frac{Bi_1}{\beta^2_{X32,m} + Bi_1^2}\right)}$	See Eq. (15d).

*The $X22$ case has the special eigenvalue of zero which is treated by the Δ_{XIJ} term in Eq. (16).

The index m is for $m = 1, 2, \ldots$.

This equation shows that $S^{(0)}_{XIJ}(x)$ is one-half of the steady-state solution. As a consequence, $S^{(0)}_{XIJ}(x)$ is a solution of the steady-state heat conduction equation in a plate,

$$\frac{d^2 S^{(0)}_{XIJ}(x)}{dx^2} = 0, \quad (n = 0) \tag{18b}$$

with appropriate boundary conditions. A general solution of Eq. (18b) (i.e., for $n = 0$) is

$$S^{(0)}_{XIJ}(x) = C^{(0)}_{XIJ,1} x + C^{(0)}_{XIJ,2}, \quad (n = 0) \tag{18c}$$

with the constants found using the given boundary conditions for the temperature problem.

For the linear-in-time variation, (i.e., $n = 1$ in Eq. (6d)), the temperature for all XIJ cases (except $X22$) and for large times is obtained from Eq. (16b) as

$$T^{(1)}_{XIJ}(x,t)\Big|_{t\,\mathrm{large}} = 2tS^{(0)}_{XIJ}(x) - 2S^{(1)}_{XIJ}(x) \tag{19a}$$

Now Eq. (19a) must satisfy the transient heat conduction equation, Eq. (6d), to yield

$$2t\frac{d^2 S^{(0)}_{XIJ}}{dx^2} - 2\frac{d^2 S^{(1)}_{XIJ}}{dx^2} = 2S^{(0)}_{XIJ} \tag{19b}$$

Using Eq. (18b) causes the time-dependent term which is the first one in Eq. (19b) to disappear. (This type of simplification also occurs as n is increased to 2 and 3.) Then Eq. (19b) becomes

$$-\frac{d^2 S^{(1)}_{XIJ}}{dx^2} = S^{(0)}_{XIJ}(x) \tag{19c}$$

which can be integrated twice to obtain

$$S_{XIJ}^{(1)}(x) = -\int_{x'=0}^{x}\int_{x''=0}^{x'} S_{XIJ}^{(0)}(x'')\mathrm{d}x''\,\mathrm{d}x' + C_{XIJ,1}^{(1)}x + C_{XIJ,2}^{(1)} \tag{20a}$$

Fortuitously this equation can be extended to obtain the recursion relation,

$$S_{XIJ}^{(i)}(x) = -\int_{x'=0}^{x}\int_{x''=0}^{x'} S_{XIJ}^{(i-1)}(x'')\mathrm{d}x''\,\mathrm{d}x' + C_{XIJ,1}^{(i)}x + C_{XIJ,2}^{(i)},\quad i = 1, 2 \text{ or } 3 \tag{20b}$$

which is valid for the eight cases: I and $J = 1$, 2 or 3 except $I = J = 2$. Before finding the two constants in Eq. (20b), boundary conditions must be selected. To shorten the notation, let an I before the S denote an integration with respect to the spatial coordinate so that

$$IIS_{XIJ}^{(i-1)}(x) \equiv \int_{x'=0}^{x}\int_{x''=0}^{x'} S_{XIJ}^{(i-1)}(x'')\mathrm{d}x''\,\mathrm{d}x' \tag{20c}$$

which allows Eq. (20b) to be written as

$$S_{XIJ}^{(i)}(x) = -IIS_{XIJ}^{(i-1)}(x) + C_{XIJ,1}^{(i)}x + C_{XIJ,2}^{(i)} \tag{20d}$$

The constants in Eq. (20d) can be considered in two groups. The first group includes the $X11$, $X12$, $X21$ and $X22$ cases, which are discussed in Sections 4.1 and 4.2. For this group (including the anomalous case of $X22$), the summations are known (see Appendix A) and thus are not derived in this paper. The other five cases of $X13$, $X23$, $X31$, $X32$ and $X33$ do not have known algebraic forms for the S functions. However, the S recursion relations can be found for all of these cases at once (with the possible exception of the $X23$ case) by solving the $X33$ case, which is the most general. For this reason, the most complete derivation is for the $X33$ case, which is given in Section 4.3.

4. Algebraic solutions for the summations and temperatures

4.1. Temperature specified at $x = 0$, $X1JB3n0T0$, $J = 1$ or 2

Detailed solutions for the temperature varying as t^n at $x = 0$ surface are now discussed. The cases are denoted $X11B3n0T0$ and $X12B3n0T0$ where n can be either 0, 1, 2 or 3. The notation for a linear variation in time at $x = 0$ is given in [2, p. 28] as $X1JB20T0$. In this paper, let us use $X1JB3n0T0$ ($n = 1$) for the linear variation case, $X1JB3n0T0$ ($n = 2$) for the quadratic variation, and $X1JB3n0T0$ ($n = 3$) for the cubic variation.

Consider now the $J = 1$ case, which is for zero temperature at $x = 1$. Then the S summations defined by Eq. (15e) can be written asf

$$S_{X11}^{(i)} = \sum_{m=1}^{\infty} A_{X11,m}\frac{X_{X11,m}(x)}{\beta_{X11,m}^{2i}} = \sum_{m=1}^{\infty}\frac{1}{\beta_{X11,m}}\frac{\sin(\beta_{X11,m}x)}{\beta_{X11,m}^{2i}} = \sum_{m=1}^{\infty}\frac{\sin(m\pi x)}{(m\pi)^{2i+1}} \tag{21}$$

where the eigenfunctions, etc. are listed in Table 1, third column and second through fourth rows. For $i = 0$, 1, 2, 3, ..., this series is known and can be expressed in an algebraic form in terms of Bernoulli polynomials; see Eq. (A.2) in Appendix A. Explicit results for $i = 0$, 1, 2 and 3 for Eq. (21) are given in Table 2, column 2. For example, the temperature for the $X11B3n0T0$ ($n = 1$) case using Eq. (16b) and Table 2 is

$$T_{X11}^{(1)}(x,t) = 2tS_{X11}^{(0)}(x) - 2S_{X11}^{(1)}(x) + 2\sum_{m=1}^{\infty}\mathrm{e}^{-(m\pi)^2t}\frac{\sin(m\pi x)}{(m\pi)^3} = t(1-x) - \frac{x}{6}(2-3x+x^2) + 2\sum_{m=1}^{\infty}\mathrm{e}^{-(m\pi)^2t}\frac{\sin(m\pi x)}{(m\pi)^3} \tag{22}$$

The $X12$ case can be similarly treated because another identity is available. In this case, analogous to Eq. (21), the summation is

$$S_{X12}^{(i)} = \sum_{m=1}^{\infty} A_{X12,m}\frac{X_{X12,m}(x)}{\beta_{X12,m}^{2i}} = \sum_{m=1}^{\infty}\frac{\sin((m-1/2)\pi x)}{((m-1/2)\pi)^{2i+1}},\quad i = 0, 1, 2, \ldots \tag{23}$$

which is related to the Euler polynomials in Eq. (A.4); the algebraic forms are given in Table 2, column 3. For linear variation with respect to t in the surface temperature, (case $X12B3n0T0$ ($n = 1$)), the temperature analogous to Eq. (22) is

$$T_{X12}^{(1)}(x,t) = t - \frac{x}{2}(2-x) + 2\sum_{m=1}^{\infty}\mathrm{e}^{-(m\pi)^2t}\frac{\sin((m-1/2)\pi x)}{((m-1/2)\pi)^3} \tag{24}$$

and the $n = 2$ and $n = 3$ solutions can be similarly written using Eqs. (16c,d) and Table 2.

For the convective boundary condition at $x = 1$, series analogous to the Bernoulli and Euler polynomials are not available, but the $X13$ results can be obtained from the general case of $X33$. See Section 4.3.

4.2. Heat flux specified at $x = 0$, $X2JB3n0T0$, $J = 1$ or 2

For the $X2JB3n0T0$ series of cases, as the previous subsection, the S functions can be expressed in terms of Bernoulli and Euler polynomials for $J = 1$ or 2. The values are given in columns 4 and 5 of Table 2. The results for the $X22B3n0T0$ series are different because $\Delta_{X22} = 1$. Dimensionless temperatures for the $X22B3n0T0$ ($n = 0$, 1 or 2) cases are

$$T^{(0)}(x,t) = t + \frac{2-6x+3x^2}{6} - 2\sum_{m=1}^{\infty}\mathrm{e}^{-(m\pi)^2t}\frac{\cos(m\pi x)}{(m\pi)^2} \tag{25a}$$

$$T^{(1)}(x,t) = \frac{t^2}{2} + t\frac{2-6x+3x^2}{6} - \frac{(8-60x^2+60x^3-15x^4)}{360} + 2\sum_{m=1}^{\infty}\mathrm{e}^{-(m\pi)^2t}\frac{\cos(m\pi x)}{(m\pi)^4} \tag{25b}$$

Table 2
Algebraic forms for $S_{XIJ}^{(i)}(x)$ summations for I and $J = 1$ and 2

	$X11$ case	$X12$ case	$X21$ case	$X22$ case
$S_{XIJ}^{(i)}(x)$	$S_{X11}^{(i)}(x) = \sum_{m=1}^{\infty} \frac{\sin(\beta_{X11,m}x)}{\beta_{X11,m}^{2i+1}}$ $\beta_{X11,m} = m\pi$	$S_{X12}^{(i)}(x) = \sum_{m=1}^{\infty} \frac{\sin(\beta_{X12,m}x)}{\beta_{X12,m}^{2i+1}}$ $\beta_{X12,m} = (m-1/2)\pi$	$S_{X21}^{(i)}(x) = \sum_{m=1}^{\infty} \frac{\cos(\beta_{X21,m}x)}{\beta_{X21,m}^{2(i+1)}}$ $\beta_{X21,m} = (m-1/2)\pi$	$S_{X22}^{(i)}(x) = \sum_{m=1}^{\infty} \frac{\cos(\beta_{X22,m}x)}{\beta_{X22,m}^{2(i+1)}}$ $\beta_{X22,m} = m\pi$
$i = 0$	$S_{X11}^{(0)}(x) = \frac{1}{2}(1-x)$	$S_{X12}^{(0)}(x) = \frac{1}{2}$	$S_{X21}^{(0)}(x) = \frac{1}{2}(1-x)$	$S_{X22}^{(0)}(x) = \frac{2-6x+3x^2}{12}$
$i = 1$	$S_{X11}^{(1)}(x) = \frac{x}{12}(2-3x+x^2)$	$S_{X12}^{(1)}(x) = \frac{x(2-x)}{4}$	$S_{X21}^{(1)}(x) = \frac{2-3x^2+x^3}{12}$	$\frac{(8-60x^2+60x^3-15x^4)}{720}$
$i = 2$	$S_{X11}^{(2)}(x) = \frac{x(8-20x^2+15x^3-3x^4)}{720}$	$S_{X12}^{(2)}(x) = \frac{x(8-4x^2+x^3)}{48}$	$\frac{(16-20x^2+5x^4-x^5)}{240}$	$\frac{(32-168x^2+210x^4-126x^5+21x^6)}{30240}$
$i = 3$	$\frac{x(32-56x^2+42x^4-21x^5+3x^6)}{30240}$	$\frac{x(96-40x^2+6x^4-x^5)}{1440}$	$\frac{(272-336x^2+70x^4-7x^6+x^7)}{10080}$	$\frac{(128-640x^2+560x^4-280x^6+120x^7-15x^8)}{1209600}$

Power variations are for t^n ($n = 0$, 1, 2 and 3) for the temperature or heat flux at $x = 0$.

$$T^{(2)}(x,t) = \frac{t^3}{3} + t^2\frac{2-6x+3x^2}{6} - t\frac{8-60x^2+60x^3-15x^4}{180} + \frac{32-168x^2+210x^4-126x^5+21x^6}{7560} - 4\sum_{m=1}^{\infty} e^{-(m\pi)^2 t}\frac{\cos(m\pi x)}{(m\pi)^6} \tag{25c}$$

The $X23B3n0T0$ ($n = 1$, 2 or 3) series of cases is discussed in the next subsection.

4.3. Ambient temperature specified at $x = 0$, $X33Bn0T0$ cases

In this subsection, the general case of the $X33B3n0T0$ is derived and is the basis for also treating the $X13B3n0T0$, $X23B3n0T0$, $X31B3n0T0$ and $X32B3n0T0$ cases. Using Eqs. (15d) and (15e) and the $X_{X33,m}(x)$ function given in Table 1, the $S_{X33}^{(i)}(x)$ summation is

$$S_{X33}^{(i)}(x) = Bi_1\sum_{m=1}^{\infty}\frac{\beta_{X33,m}\cos(\beta_{X33,m}x) + Bi_1\sin(\beta_{X33,m}x)}{\left(\beta_{X33,m}^2 + Bi_1^2\right)[1 + Bi_2/(\beta_{X33,m}^2 + Bi_2^2)] + Bi_1} \times \frac{1}{\beta_{X33,m}^{2i+1}} \tag{26a}$$

Eq. (26a) can be used for the $X1J$ and $X3J, J = 1$, 2 or 3 cases since the permissible range of Bi_1 is $0 < Bi_1 \leqslant \infty$. Note that this excludes the case of $Bi_1 = 0$ (i.e., the $X2J$ cases) but the case of $X23$ can be accommodated as

$$S_{X23}^{(i)}(x) = \sum_{m=1}^{\infty}\frac{\cos(\beta_{X23,m}x)}{\beta_{X23,m}[1 + Bi_2/(\beta_{X23,m}^2 + Bi_2^2)]}\frac{1}{\beta_{X23,m}^{2i+1}} \tag{26b}$$

The derivation for the $X33$ case starts with Eq. (20d) and the two boundary conditions which are given in Eqs. (8b) and (9b) and repeated here as,

$$-\frac{\partial T_{X33}^{(n)}}{\partial x}(0,t) = Bi_1(t^n - T_{X33}^{(n)}(0,t)), \quad n = 0,1,2 \text{ or } 3 \quad (27a)$$

$$-\frac{\partial T_{X33}^{(n)}}{\partial x}(1,t) = Bi_2 T_{X33}^{(n)}(1,t), \quad n = 0,1,2 \text{ or } 3 \quad (27b)$$

For a constant ambient temperature condition, $n = 0$ in Eq. (27a) and then $t^n = 1$.Then the steady state temperature solution for these boundary conditions ($X33B10$) is

$$T^{(0)}(x) = \frac{Bi_1[1 + Bi_2(1-x)]}{[Bi_1 + Bi_2(1 + Bi_1)]} \quad (28)$$

Then the S function for $i = 0$ (1/2 of Eq. (28) as indicated below Eq. (17a)) is

$$S_{X33}^{(0)}(x) = \frac{Bi_1[1 + Bi_2(1-x)]}{2[Bi_1 + Bi_2(1 + Bi_1)]} \quad (29)$$

This equation is valid for all values of the Biot numbers except $Bi_1 = 0$. It can be used for $Bi_2 = 0$ for which the value of 1/2 is obtained. Hence,

$$S_{X12}^{(0)}(x) = \frac{1}{2} \quad (30a)$$

$$S_{X32}^{(0)}(x) = \frac{1}{2} \quad (30b)$$

(Notice the $X22$ case is not included.) Eq. (29) can also be used for $Bi_1 \to \infty$ to obtain

$$S_{X11}^{(0)}(x) = \frac{1-x}{2} \quad (31a)$$

$$S_{X13}^{(0)}(x) = \frac{1 + Bi_2(1-x)}{2(1 + Bi_2)} \quad (31b)$$

Finally, the condition of $Bi_2 \to \infty$ yields

$$S_{X31}^{(0)}(x) = \frac{Bi_1(1-x)}{2(1 + Bi_1)} \quad (32)$$

This covers six of the 9 cases. The three remaining cases are $X21$, $X22$ and $X23$. The first two of these cases are covered in Table 2, columns 4 and 5, third row. The $X23$ result is

$$S_{X23}^{(0)}(x) = \frac{1 + Bi_2(1-x)}{2Bi_2} \quad (33)$$

Recursive relations for the S-functions are now derived. Start with Eq. (16a) (which is for $n = 0$) for the temperature for large t's and drop the $\Delta_{X33} = 0$ term to get

$$T_{X33}^{(0)}(x,t)\Big|_{t\,\text{large}} = 2S_{X33}^{(0)}(x) \quad (34)$$

Introducing this equation in Eqs. (27a,b) for $n = 0$ gives

$$-2\frac{dS_{X33}^{(0)}}{dx}(0) = Bi_1(1 - 2S_{X33}^{(0)}(0)) \quad (35a)$$

$$-2\frac{dS_{X33}^{(0)}}{dx}(1) = Bi_2 2S_{X33}^{(0)}(1) \quad (35b)$$

Now use Eq. (16b) (which is for $n = 1$) for large t to get

$$T_{X33}^{(1)}(x,t)\Big|_{t\,\text{large}} = 2tS_{X33}^{(0)}(x) - 2S_{X33}^{(1)}(x) \quad (36)$$

Introduce Eq. (36) into Eq. (27a) with $n = 1$ to find

$$-2t\frac{dS_{X33}^{(0)}}{dx}(0) + 2\frac{dS_{X33}^{(1)}}{dx}(0) = Bi_1(t - 2tS_{X33}^{(0)}(0) + 2S_{X33}^{(1)}(0)) \quad (37a)$$

Re-arrange this equation so that the function of t is on the right side,

$$2\frac{dS_{X33}^{(1)}}{dx}(0) - 2Bi_1 S_{X33}^{(1)}(0) = t\left(Bi_1(1 - 2S_{X33}^{(0)}(0)) + 2\frac{dS_{X33}^{(0)}}{dx}(0)\right) \quad (37b)$$

Using Eqs. (35b) and (35b) shows that the right side of this equation is equal to zero. Consequently Eq. (37b) can be written as

$$\frac{dS_{X33}^{(1)}}{dx}(0) = Bi_1 S_{X33}^{(1)}(0) \quad (38)$$

Repeat the same procedure by introducing Eq. (36) into Eq. (27b) with $n = 1$ to find

$$-2t\frac{dS_{X33}^{(0)}}{dx}(1) + 2\frac{dS_{X33}^{(1)}}{dx}(1) = Bi_2\left(2tS_{X33}^{(0)}(1) - 2S_{X33}^{(1)}(1)\right) \quad (39b)$$

$$\frac{dS_{X33}^{(1)}}{dx}(1) + Bi_2 S_{X33}^{(1)}(1) = t\left(\frac{dS_{X33}^{(0)}}{dx}(1) + Bi_2 S_{X33}^{(0)}(1)\right) \quad (39c)$$

Again the right side is seen to be zero when it is compared with Eq. (35b). Hence

$$\frac{dS_{X33}^{(1)}}{dx}(1) = -Bi_2 S_{X33}^{(1)}(1) \quad (40)$$

At this point the two boundary conditions are used, resulting in Eqs. (38) and (40). Use the general equation for S, Eq. (20d), for $i = 1$ and $XIJ = X33$ to obtain

$$S_{X33}^{(1)}(x) = -IIS_{X33}^{(0)}(x) + C_{X33,1}^{(1)}x + C_{X33,2}^{(1)} \quad (41)$$

This equation is introduced into Eq. (38) to find

$$-\frac{dIIS_{X33}^{(0)}}{dx}(0) + C_{X33,1}^{(1)} = Bi_1(-IIS_{X33}^{(0)}(0) + C_{X33,2}^{(1)}) \quad (42a)$$

The terms related to IIS need to be carefully examined. Recall the definition given by Eq. (20c) which when written for this case becomes

$$IIS_{X33}^{(1)}(x) \equiv \int_{x'=0}^{x} \int_{x''=0}^{x'} S_{X33}^{(0)}(x'')\,dx''\,dx' \quad (42b)$$

When the upper limit of the outer integration is set equal to zero, this double integral is zero,

$$IIS_{X33}^{(1)}(0) = 0 \quad (42c)$$

which is valid for $i = 1$ and for i equal to all positive integers. Taking the first derivative with respect to x of Eq. (42b) and using Liebnitz' rule for differentiation of an integral yields

$$\frac{\mathrm{d}IIS_{X33}^{(1)}}{\mathrm{d}x}(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x}\int_{x'=0}^{x}\int_{x''=0}^{x'} S_{X33}^{(0)}(x'')\mathrm{d}x''\,\mathrm{d}x' = \int_{x''=0}^{x} S_{X33}^{(0)}(x'')\mathrm{d}x'' = IS_{X33}^{(0)}(x) \tag{42d}$$

which is also equal to zero at $x=0$. Hence, Eq. (42a) becomes simply

$$C_{X33,1}^{(1)} = Bi_1 C_{X33,2}^{(1)} \tag{42e}$$

Now consider the boundary condition at $x=1$ by introducing Eq. (41) into Eq. (40),

$$-IS_{X33}^{(0)}(1) + C_{X33,1}^{(1)} = -Bi_2\left(-IIS_{X33}^{(0)}(1) + C_{X33,1}^{(1)} + C_{X33,2}^{(1)}\right) \tag{43}$$

Two algebraic equations, Eqs. (42e) and (43), are available for the two unknowns, $C_{X33,1}^{(1)}$ and $C_{X33,2}^{(1)}$. It is also true that Eqs. (42e) and (43) apply for $i=2$ and 3. Using the solution for the two unknowns and then introducing them back into Eq. (41), (generalized for $i=1$, 2 or 3) gives for the $X33B3n0T0$ ($n=1$, 2 or 3) series of S's the recursion relation,

$$S_{X33}^{(i)}(x) = -IIS_{X33}^{(i-1)}(x) + \frac{Bi_1 x + 1}{Bi_1 + Bi_2 + Bi_1 Bi_2}\left[IS_{X33}^{(i-1)}(1) + Bi_2 IIS_{X33}^{(i-1)}(1)\right] \tag{44}$$

This recursion relation is general and can be used to obtain all the cases for XIJ, for $I=1$, 2 or 3 and $J=1$, 2 or 3, except for $I=J=2$. (However, the previously considered $X11$, $X12$, $X21$ and $X22$ cases are tabulated in Table 2.) For the boundary condition of the first kind, the Biot number is made to approach infinity; specifically, for $I=1$, $Bi_1 \to \infty$ and for $J=1, Bi_2 \to \infty$. For the boundary condition of the second kind, the Biot number is set equal to zero.

More explicitly the recursion relation for the S series for $X13B3n0T0$ ($n=1$, 2, 3) is obtained by letting $Bi_1 \to \infty$,

$$S_{X13}^{(i)}(x) = -IIS_{X13}^{(i-1)}(x) + \frac{x}{Bi_2+1}\left(IS_{X13}^{(i-1)}(1) + Bi_2 IIS_{X13}^{(i-1)}(1)\right) \tag{45}$$

Detailed results are given in Table 3, second column. For the $X23B3n0T0$ series of S's, the recursion relation for $Bi_1 \to 0$ is (see Table 3, third column)

$$S_{X23}^{(i)}(x) = -IIS_{X23}^{(i-1)}(x) + IIS_{X23}^{(i-1)}(1) + \frac{1}{Bi_2} IS_{X23}^{(i-1)}(1) \tag{46}$$

Table 3
Algebraic forms for the $S_{X13}^{(i)}(x)$ and $S_{X23}^{(i)}(x)$ summations

i	$S_{X13}^{(i)}(x) = \sum_{m=1}^{\infty}\frac{(\beta_m^2+Bi_2^2)\sin[\beta_m x]}{\beta_m^{2i+1}(\beta_m^2+Bi_2^2+Bi_2)}$	$S_{X23}^{(i)}(x) = \sum_{m=1}^{\infty}\frac{(\beta_m^2+Bi_2^2)\cos(\beta_m x)}{\beta_m^{2(i+1)}(\beta_m^2+Bi_2^2+Bi_2)}$
0	$S_{X13}^{(0)}(x) = \frac{1+Bi_2(1-x)}{2(1+Bi_2)}$	$S_{X23}^{(0)}(x) = \frac{1+Bi_2(1-x)}{2Bi_2}$
1	$S_{X13}^{(1)}(x) = \frac{x\left\{\begin{matrix}6-3x+Bi_2(6-6x+x^2)\\+Bi_2^2(2-3x+x^2)\end{matrix}\right\}}{12(1+Bi_2)^2}$	$\frac{\left[\begin{matrix}6+3Bi_2(2-x^2)+\\Bi_2^2(2-3x^2+x^3)\end{matrix}\right]}{12Bi_2^2}$
2	$\frac{x\left\{\begin{matrix}15(8-4x^2+x^3)\\-3Bi_2(-40+40x^2-15x^3+x^4)\\+Bi_2^2(48-80x^2+45x^3-6x^4)\\+Bi_2^3(8-20x^2+15x^3-3x^4)\end{matrix}\right\}}{720(1+Bi_2)^3}$	$\frac{\left[\begin{matrix}120+Bi_2(160-60x^2)+\\5Bi_2^2(16-12x^2+x^4)+\\Bi_2^3(16-20x^2+5x^4-x^5)\end{matrix}\right]}{240Bi_2^3}$
3	$\frac{x\left\{\begin{matrix}21(96-40x^2+6x^4-x^5)\\+3Bi_2(784-560x^2+126x^4-28x^5+x^6)\\+3Bi_2^2(384-392x^2+140x^4-42x^5+3x^6)\\+Bi_2^3(288-392x^2+210x^4-84x^5+9x^6)\\+Bi_2^4(32-56x^2+42x^4-21x^5+3x^6)\end{matrix}\right\}}{30240(1+Bi_2)^4}$	$\frac{\left[\begin{matrix}5040+840Bi_2(10-3x^2)\\+42Bi_2^2(136-80x^2+5x^4)\\+7Bi_2^3\begin{pmatrix}272-240x^2\\+30x^4-x^6\end{pmatrix}\\+Bi_2^4\begin{pmatrix}272-336x^2\\+70x^4-7x^6+x^7\end{pmatrix}\end{matrix}\right]}{10080Bi_2^4}$

Power variations are t^n, ($n=0$, 1, 2 and 3) for the surface temperature or heat flux at $x=0$ with a homogeneous boundary condition of the third kind at $x=1$.

The recursion relation for the $X31B3n0T0$ series of S's is ($Bi_2 \to \infty$)

$$S^{(i)}_{X31}(x) = -IIS^{(i-1)}_{X31}(x) + \frac{Bi_1 x + 1}{Bi_1 + 1} IIS^{(i-1)}_{X31}(1) \tag{47}$$

For the $X32B3n0T0$ series of S's, the recursion relation is ($Bi_2 \to 0$)

$$S^{(i)}_{X32}(x) = -IIS^{(i-1)}_{X32}(x) + \frac{Bi_1 x + 1}{Bi_1} IS^{(i-1)}_{X32}(1) \tag{48}$$

The expression for $S^{(i)}_{X23}(x)$ given by Eq. (46) is developed starting from Eq. (33). Likewise, Eq. (45) for $S^{(i)}_{X13}(x)$ is developed from (31b) as are $S^{(i)}_{X31}(x)$ and $S^{(i)}_{X32}(x)$ starting with Eqs. (32) and (30b), respectively.

5. Numerical values, graphical results and insights

Numerical values are displayed in Table 4 for the case of a cubic variation of the surface heat flux. Results are given for the $X21B3n0T0$, $X23B3n0T0$ (with $Bi_2 = 1$) and $X22B3n0T0$ cases, each with $n = 3$. Sufficient numbers of terms are used in the time-dependent, infinite series to obtain eight decimal place accuracy; the largest number of terms is for the smallest time considered which is $t = 0.05$. This small dimensionless time is chosen for several reasons. Significant penetration of the temperature does not reach the $x = 1$ side at this time and because the only difference between these cases is the boundary condition at the $x = 1$, the numerical values of these temperatures should be the same, as they are. Another important reason is that a temperatures of zero at $x = 0.75$ is useful in demonstrating intrinsic verification [8]. Consider Eq. (25c) which is for the $X22B3n0T0$ ($n = 2$) case. For $t \leqslant 0.05$ and at $x \geqslant 0.75$ and larger, the temperature is zero (to eight decimal places), but none of the terms in Eq. (25c) is zero. Hence, there must be precise canceling of terms, which is an indication of intrinsic verification [8]. In general, close subtraction of two large numbers is inadvisable, but for exact solutions it may help to amplify (and thus reveal) errors in the equations or the programming of them.

For small dimensionless times, indicated by incomplete temperature penetration, the solution reduces to that given for a semi-infinite body. For example, the surface temperature for a heat flux varying as t^3 obtained from Eq. (1b) is

$$T^{(3)}_{X20}(0,t) = \frac{32}{35\sqrt{\pi}} t^{7/2} \tag{49}$$

which gives the values of 0.00001442 and 0.00402993 for $t = 0.05$ and 0.25, respectively. The first number is exactly the same as in Table 4; the second number differs only in the last two digits for the $X23B3n0T0$ ($n = 3$) case. Consistency with results for a semi-infinite body for small dimensionless time is another indication of intrinsic verification [8].

Table 4 also shows that the $X21B3n0T0$ ($n = 3$) and $X22B3n0T0$ ($n = 3$) cases span the extremes for the $X23B3n0T0$ ($n = 3$) for Biot numbers from zero to infinity. The curves for this case would be between the $X21B3n0T0$ ($n = 3$) and $X22B3n0T0$ ($n = 3$) cases, which are shown in Fig. 1, but the $X23B3n0T0$ ($n = 3$) curve is not shown because it is obscured by the forgoing curves. Cases for $n = 0$ and 1 are also shown in Fig. 1.

The S terms in Table 3 for the $X13$ case includes coefficients for the $Bi_2 = 0$ and $Bi_2 = \infty$ cases. For the first of

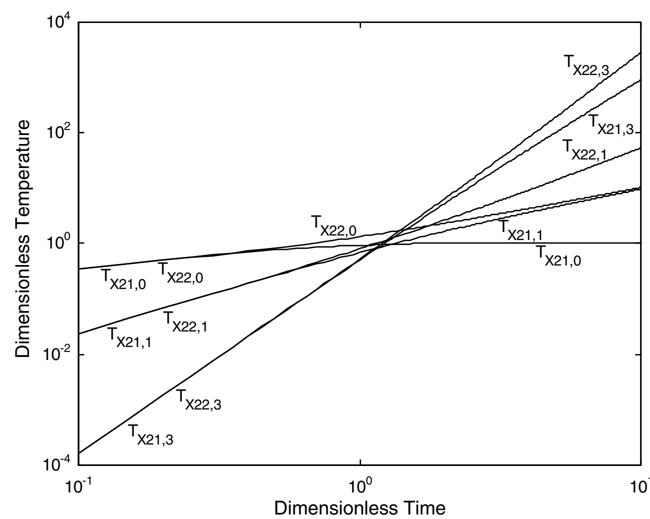

Fig. 1. Dimensionless temperatures for the $X21B3n0T0$ ($n = 0$, 1,3) cases (which are labeled $T_{X21,0}$, $T_{X21,1}$ and $T_{X21,3}$, respectively.) Also shown are the $X22B3n0T0$ ($n = 0$, 1,3) cases, labeled $T_{X22,0}$, $T_{X22,1}$ and $T_{X22,3}$.

Table 4
Numerical values for cubic ($n = 3$) variations, $X21B3n0T0$, $X23B3n0T0$ (with $B_2 = 1$) and $X22B3n0T0$ cases

Case	t	# terms	$T(0,t)$	$T(0.25,t)$	$T(0.5,t)$	$T(0.75,t)$	$T(1,t)$
$X21$	0.05	4	0.00001442	0.00000139	0.00000009	0.00000000	0.00000000
$X23$	0.05	4	0.00001442	0.00000139	0.00000009	0.00000000	0.00000000
$X22$	0.05	4	0.00001442	0.00000139	0.00000009	0.00000000	0.00000000
$X21$	0.25	2	0.00402961	0.00147840	0.00050370	0.00014860	0.00000000
$X23$	0.25	2	0.00403016	0.00147975	0.00050930	0.00017068	0.00008019
$X22$	0.25	2	0.00403024	0.00147997	0.00051025	0.00017466	0.00009550
$X21$	1.00	1	0.50364687	0.30059855	0.16708016	0.07420347	0.00000000
$X23$	1.00	1	0.52128516	0.32153035	0.19883803	0.12734487	0.09077119
$X22$	1.00	1	0.52809530	0.32975886	0.21179188	0.14985757	0.13062493
$X21$	2.00	1	5.27851892	3.52362358	2.15087451	1.01776124	0.00000000
$X23$	2.00	1	5.98720896	4.28686694	3.08400770	2.25526434	1.70978408
$X22$	2.00	1	6.42412698	4.76242996	3.68087050	3.07271431	2.87665675

these, the case becomes $X12$ and the equations collapse to those given in Table 2, third column and the other case becomes $X11$ given in the second column of Table 2.

It is instructive to examine plots of the S functions because they determine the variation with x of temperature as t becomes large. Furthermore, insight can be gained by examining them. See Fig. 2 for scaled values for $S_{X11}^{(i)}(x)$ and $S_{X22}^{(i)}(x)$ for $i = 0$, 1, 2 and 3. Scaling allows the values to be conveniently displayed in the same plot and provides insight for a larger set of problems. The physical values of S are found by multiplying S by the scaling coefficients given in the heading of Fig. 2. The $i = 0$ curves are noticeably different in shape from the $i = 1$, 2 and 3 curves, with the latter two being very nearly the same; see, for example, the scaled $S_{X11}^{(i)}(x)$ curves. For $i = 0$, the scaled $S_{X11}^{(0)}(x)$, shown by dots, decreases linearly from one to zero while the other S-curves approach what seems to be part of a sine curve. In fact, the scaled $S_{X11}^{(i)}(x)$ curves approach $\sin(\pi x)$ as i increases.

Fig. 3 shows the scaled $S_{X12}^{(i)}(x)$ and $S_{X21}^{(i)}(x)$ curves. Notice that the scaled $S_{X12}^{(0)}(x)$, denoted by dots, has a constant value of unity. The previously observed behavior of approaching common results is seen for increasing i. This hints that the S-series can be approximated by just a few terms when i increases to 2 or 3.

The algebraic forms tend to become unwieldy as i increases, particularly for boundary conditions of the third kind, as indicated in Table 3 for $i = 2$ and 3. However, at the same time, the required number of terms in the quasi-steady summations is reduced. An alternate way to evaluate the temperature solution for the $X23B3n0T0$ ($n = 3$) problem is to use algebraic identities for $i = 0$ and 1 and the series representation for $i = 2$ and 3, as in

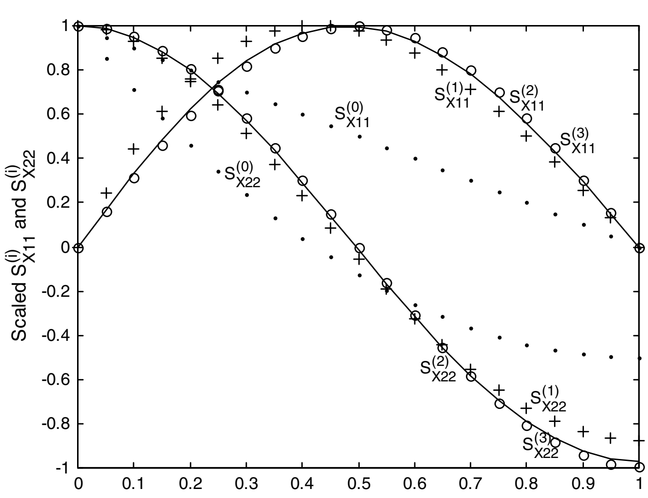

Fig. 2. Scaled values for $S_{X11}^{(i)}(x)$ and $S_{X22}^{(i)}(x)$ for $i = 0$ (denoted by a ●), $i = 1$ (denoted by a +), $i = 2$ (denoted by a continuous line) and $i = 3$ (denoted by an ○). The scaling factors for $S_{X11}^{(i)}(x)$ for $i = 0$, 1, 2 and 3 are 0.5, 0.0320, 0.00326 and 0.000331, respectively. The scaling factors for $S_{X22}^{(i)}(x)$ for $i = 0$, 1, 2 and 3 are 0.1667, 0.0111, 0.001058 and 0.0001058, respectively.

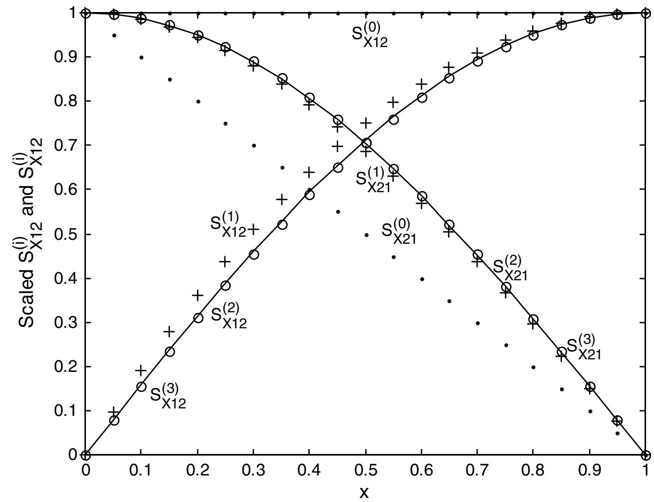

Fig. 3. Scaled values for $S_{X12}^{(i)}(x)$ and $S_{X21}^{(i)}(x)$ for $i = 0$ (denoted by a ●), $i = 1$ (denoted by a +), $i = 2$ (denoted by a continuous line) and $i = 3$ (denoted by a ○). The scaling factors for $S_{X12,i}(x)$ for $i = 0$, 1, 2 and 3 are 0.5, 0.25, 0.1042 and 0.0424, respectively. The scaling factors for $S_{X21,i}(x)$ for $i = 0$, 1, 2 and 3 are 0.5, 0.1667, 0.0667 and 0.0270, respectively.

$$T_{X23}^{(3)}(x,t) = 2t^3 S_{X23}^{(0)}(x) - 6t^2 S_{X23}^{(1)}(x) + 12\sum_{m=1}^{M}\left\{\frac{(\beta_{X23,m}^2 + Bi_2^2)\cos(\beta_{X23,m}x)}{\beta_{X23,m}^6(\beta_{X23,m}^2 + Bi_2^2 + Bi_2)}\right\} \times \left[t - \frac{1}{\beta_{X23,m}^2}\left(1 - e^{-\beta_{X23,m}^2 t}\right)\right] \tag{50}$$

where $S_{X23}^{(0)}(x)$ and $S_{X23}^{(1)}(x)$ are given in Table 3, column 3. The fifth column, fifth second row of Table 1 lists $\beta_{X23,m}$. The number of terms in the summation, M, need not be large in Eq. (50). For example, for $Bi_2 = 1$ and $t = 0.25$, the temperature at $x = 0$ is 0.0028, 0.00399, 0.004025, 0.004029 and 0.00403015, for $M = 1$, 2, 3, 4 and 10, respectively. A more accurate value using the algebraic values of $S_{X23}^{(i)}(x)$ for $i = 0$ to 3 is given in Table 2 as 0.00403016, for which only two terms were needed in its summation. However, the summation given by Eq. (50) may be easier to use, particularly if only 1% accuracy is needed since only two terms are then needed. For larger values of t, even fewer terms are needed. For $x = 0$, $t = 2$ and $Bi_2 = 1$, the computed temperatures are 5.97, 5.987, 5.98717, 5.987201 and 5.98720889 for $M = 1$, 2, 3, 4 and 10, respectively. The last two digits are 96 in Table 4 instead of 89 in the last number just given. Only one term is needed in the series in Eq. (50) to obtain accuracies of better than 0.3% for $x = 0$ and t values equal to or greater than 2. As a consequence of the rapid convergence of Eq. (50), rarely would more than 10 terms be needed in the summation.

6. Summary and conclusions

Solutions are given for the temperature response in finite plates subject to time varying boundary conditions proportional to t^n for $n = 0$, 1, 2, and 3 at the $x = 0$ surface. The boundary condition at $x = 1$ is homogeneous although this can be changed by a simple coordinate transformation. Nine combinations of boundary conditions, either first, second and third kind at each surface, are examined. A systematic protocol is given for developing algebraic forms to replace infinite series, thereby speeding computation of the temperature profile. The protocol yields steady-state terms, terms increasing as a power of time and exponentially decaying terms. For the constant surface conditions ($n = 0$), and even for the linear-in-time conditions ($n = 1$), series summations may converge slowly, particularly when the surface heat flux is needed. In contrast, the algebraic forms developed here for $n = 0$ and 1 contain only a few, easily evaluated terms. The larger powers, such as $n = 2$ and 3, introduce rapidly converging summations for $S^{(2)}_{XIJ}(x)$ and $S^{(3)}_{XIJ}(x)$. Consequently, using algebraic forms for $S^{(0)}_{XIJ}(x)$ and $S^{(1)}_{XIJ}(x)$ and summation forms for $S^{(2)}_{XIJ}(x)$ and $S^{(3)}_{XIJ}(x)$ is an efficient method of numerical evaluation for $T(x,t)$. See Eq. (50).

Others have sought to speed convergence, such as by Euler's method [2]. Texts of general methods to speed convergence may be consulted as well (e.g, Knopp [20] and Jolley [21]). The method developed here, however, seeks to replace infinite summations with equivalent algebraic expressions.

The solutions have intrinsic value because they have potential for unsteady boundary conditions approximated using splines. The solutions contain the possibility of intrinsic verification. The method given herein is sufficiently general that it can be extended to other 1D geometries in heat conduction, such as to radial and spherical coordinates.

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The insights and comments by Dr. Arafa Osman are appreciated.

Appendix A. Series relations for $S^{(i)}_{XIJ}(x)$, I and $J = 1$ and 2

The relations listed below are re-arrangements of Eqs. (19:6:6), (19:6:7), (20:6:5) and (20:6:6) in Spanier and Oldham [18]. See also Eqs. (23.1.17) and (23.1.18) in Abramowitz and Stegun [19].

$$S^{(i)}_{X22}(x) = \sum_{m=1}^{\infty} \frac{\cos(m\pi x)}{(m\pi)^{2(i+1)}} = \frac{(-1)^i 2^{2i+1}}{(2(i+1))!} B_{2(i+1)}(x/2) \quad \text{(A.1)}$$
$$i = 0, 1, 2, \ldots, 0 \leqslant x \leqslant 2$$

$$S^{(i)}_{X11}(x) = \sum_{m=1}^{\infty} \frac{\sin(m\pi x)}{(m\pi)^{2i+1}} = (-1)^{i+1} \frac{2^{2i}}{(2i+1)!} B_{2i+1}(x/2) \quad \text{(A.2)}$$
$$i = 0 : 0 < x < 2; \quad i > 0 : 0 \leqslant x \leqslant 2$$

$$S^{(i)}_{X21}(x) = \sum_{m=1}^{\infty} \frac{\cos((m-1/2)\pi x)}{((m-1/2)\pi)^{2(i+1)}} = (-1)^{i+1} \frac{2^{2i}}{(2i+1)!} E_{2i+1}(x/2)$$
$$i = 0, 1, 2, 3, \ldots \quad 0 \leqslant x \leqslant 2 \quad \text{(A.3)}$$

$$S^{(i)}_{X12}(x) = \sum_{m=1}^{\infty} \frac{\sin((m-1/2)\pi x)}{((m-1/2)\pi)^{2i+1}} = (-1)^{i} \frac{2^{2i-1}}{(2i)!} E_{2i}(x/2)$$
$$i = 0 : 0 < x < 2; \quad i > 0 : 0 \leqslant x \leqslant 2 \quad \text{(A.4)}$$

The $B_i(x/2)$ symbols are the Bernoulli polynomials, the first three of which are 1, $x/2 - 1/2$, and $(x/2)^2 - x/2 + 1/6$. The $E_i(x/2)$ symbols are called Euler polynomials, the first three of which are 1, $x/2 - 1/2$, and $(x/2)^2 - x/2$.

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